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SHORT COMMUNICATION
STABILITY OF FLOW OVER A ROTATING DISK

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SUMMARY

The perturbation equations which characterize the stability of flow over a rotating infinite disk are derived via strict order of magnitude analysis. These equations contain viscous terms not considered by Stuart,[1] curvature and Coriolis terms not considered by Brown,[2] and axial velocity terms not considered by Kobayashi *et al.*[3] The strategy for reducing the problem to an algebraic system is Galerkin's method with B-spline discretization. In comparison with the Poiseuille flow solutions of Orszag,[4] the method is shown to perform well without placing undue demands on computing capability. Critical values of Reynolds number, wave length, vortex orientation and number of spiral vortices calculated by the present method compare favourably with experimental data of Kobayashi *et al.*

KEY WORDS Galerkin Spline Stability Disk

INTRODUCTION

Stability of the laminar flow which is due to the rotation of an infinite disk has been investigated by several researchers over the past forty years. Stuart[1] examined stability at infinite Reynolds number, considering the velocity component in the direction of wave propagation to be approximately two dimensional. He found the critical profile to possess an inflection point at the position of vanishing velocity. Stuart's calculations gave good agreement with experimental data on the direction of wave propagation but overestimated the number of vortices. He then concluded that viscosity must have a considerable influence on the wave number. Brown[2] extended Stuart's analysis by considering stability at finite Reynolds number. He, nevertheless, retained the assumption of parallel basic flow. The critical Reynolds number for instability to infinitesimal disturbances obtained by Brown is $Re_c = 233$. The Reynolds number is defined as $Re = a\omega\delta/v$, where a is the radial distance from the axis of rotation and $\delta = 1{\cdot}271\,(v/\omega)^{1/2}$ is the displacement thickness. The basic flow equations contain curvature and Coriolis terms in the analysis of Kobayashi *et al.*[3], but the axial velocity component as well as the in-plane variation of the velocity are neglected. The analysis retains only terms of order Re^{-1} and larger. Kobayashi calculated 332 for the critical Reynolds number. This value together with 0·465 for critical wavelength yields $n \cong 23$ for the number of vortices. Experimentally Kobayashi *et al.* found $Re_c = 377$ at instability. This latter value of Re_c gives $n \cong 26$, which agrees with the lower end of the range 26–33 ($n = 26$ was measured with a hot-wire probe and $n = 33$ was estimated visually).

The stability equations of Malik *et al.*[5] account for both Coriolis and curvature effects to order Re^{-1}. Examining stability to perturbations that are periodic in both radial and azimuthal

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directions and time and using the Chebyshev polynomial expansion of Orszag,[4] they find $Re_c = 364{\cdot}78$ and a spiral angle $\varepsilon = 11{\cdot}2°$. Malik *et al.* map the semi-infinite domain $0 \leqslant z < \infty$ into a finite interval, prior to solution of the equations. The ambiguity associated with the positioning of the boundary at infinity (value of k in our analysis) is avoided in this manner but, at the same time, some insight into both the underlying fluid mechanics and the structure of the equations is lost.

The present analysis follows the work of Kobayashi *et al.*,[3] but differs from it in two important respects. We show that the effect of the axial velocity component on flow conditions at criticality is not negligible. We also recognize the variation of the basic flow velocity along planes parallel to the disk. The analysis yields results which compare very favourably with the experimental data of Kobayashi. For the critical values of Reynolds number, wavelength, direction of wave propagation and number of vortices we calculate 359·79, 0·486, 14° and 26, respectively. The corresponding experimental values of Kobayashi *et al.*[3] are 377·10, 0·465, 13°–15° and 26–33. The scattering in the measured value of the angle ε seems to be due to the fact that the various measurements were made at different speeds; the lower value of $\varepsilon = 13°$ representing experiments performed at lower speeds. There is also a range for the experimentally observed value of n. Apparently the spiral vortices branch off and the number n of the vortices tends to increase radially. There is also a bifurcation phenomenon of the vortices with increasing Reynolds number.[5]

The strategy employed here for numerical solution of the stability equations is Galerkin's method with B-spline discretization. The method is simple to use when employing the spline subroutine package of de Boor.[6] It is also very accurate, and is readily applicable to stability problems governed by partial differential equations.

THEORETICAL

A disk of infinite radius is located at $\bar{x}^3 = 0$ in the cylindrical polar co-ordinate system $\{\bar{x}^1, \bar{x}^2, \bar{x}^3\}$.

The velocity distribution of the flow in the half space $\bar{x}^3 > 0$, induced by rotation Ω of the disk, is defined by the Karman problem.[7] The physical components of the velocity of this basic flow relative to the $\{\bar{x}^1, \bar{x}^2, \bar{x}^3\}$ co-ordinate system are denoted by $\{U_r, V_\theta, W_z\}$.

Following Stuart,[1] we define another orthogonal curvilinear co-ordinate system, the origin of which is located on the disk at $\bar{x}^1 = a$, some $\bar{x}^2$, and which is given by the transformation:

$$T: \begin{cases} x^1 = a\left[\ln\left(\dfrac{\bar{x}^1}{a}\right)\cos\varepsilon - (\bar{x}^2 + \Omega t)\sin\varepsilon\right] \\ x^2 = a\left[\ln\left(\dfrac{\bar{x}^1}{a}\right)\sin\varepsilon + (\bar{x}^2 + \Omega t)\cos\varepsilon\right] \\ x^3 = \bar{x}^3 \end{cases} \tag{1}$$

We look for stability to perturbations which are periodic in the x^1 direction and time, are independent of x^2 and have amplitudes that are functions of x^3 alone. Let $\{\mathbf{v}; p\}$ represent the perturbation, then

$$\{\mathbf{v}(x, t); p(x, t)\} = \{\mathbf{v}(x^3); p(x^3)\}\mathrm{e}^{\mathrm{i}(\alpha x^1 - \lambda t)} \tag{2}$$

The linearized equations that govern the distribution of $\{\mathbf{v}; p\}$ are

$$-\mathrm{i}\lambda\mathbf{v} + \mathbf{V}\cdot\operatorname{grad}\mathbf{v} + \mathbf{v}\cdot\operatorname{grad}\mathbf{V} + 2\mathbf{\Omega}\times\mathbf{v} = -\operatorname{grad}\left(\frac{1}{\rho}p\right) + \nu\nabla\mathbf{v} \tag{3}$$

$$\operatorname{div}\mathbf{v} = 0 \tag{4}$$

The pressure p and the velocity component u are eliminated from equation (3) by cross-differentiation and by substitution from equation (4), respectively.

The various dimensionless quantities employed have the definition:

$$\{x^i\} = a\{x, y, \delta^1 z\}; \{U_x, V_y, W_z\} = V_0\{\bar{U}_x, \bar{V}_y, \delta^1 \bar{W}_z\}$$

$$Re = \frac{V_0\delta}{v}; \quad \sigma = \frac{\alpha\delta}{h_1}; \quad c = \frac{\lambda h_1}{\alpha V_0}; \quad \delta^1 = \frac{\delta}{a} \tag{5}$$

In (5), and elsewhere, δ is the displacement thickness of the boundary layer, $V_0 = a\Omega$ is the characteristic velocity of the problem and h_1 is the Lame coefficient and $\{U_x, V_y, W_z\}$ are the physical components of the basic flow velocity relative to $\{x^i\}$.

The boundary conditions on the perturbations are

$$u = v = w = 0, \quad \text{at} \quad z = 0 \tag{6a}$$

$$u = v = w = 0, \quad \text{at} \quad z \to \infty \tag{6b}$$

Terms of order Re^{-1} or larger are retained whereas others are neglected in the analysis.

Boundary condition (6b) suggests the introduction of a normalized co-ordinate

$$\zeta = \frac{z}{k}; \qquad 0 \leqslant \zeta \leqslant 1$$

so that the boundary conditions now assume the form:

$$v = w = \frac{\mathrm{d}w}{\mathrm{d}\zeta} = 0; \qquad \zeta = 0 \tag{7a}$$

$$v = w = \frac{\mathrm{d}w}{\mathrm{d}\zeta} = 0; \qquad \zeta = 1 \tag{7b}$$

where we took (5) into account.

Transformation of equations (3) and (4) yields

$$\begin{aligned}&(U_x - c)v + \frac{\mathrm{i}}{\bar{\sigma}Re}\left(\frac{\mathrm{d}^2 v}{\mathrm{d}\zeta^2} - \bar{\sigma}^2 v\right) - \frac{\mathrm{i}}{\bar{\sigma}}\frac{\partial V_y}{\partial \zeta}w \\ &+ \bar{\delta}^1\left\{-\frac{\mathrm{i}}{\bar{\sigma}}\left[(U_x \cos\varepsilon + V_y \sin\varepsilon)v + \frac{1}{k}W_k\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right]\right. \\ &\left.+ \frac{2}{\bar{\sigma}^2}(V_y \cos\varepsilon - U_x \sin\varepsilon - 1)\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right\} = 0\end{aligned} \tag{8a}$$

$$\begin{aligned}&(U_x - c)\left(\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} - \bar{\sigma}^2 w\right) - \frac{\partial^2 U_x}{\partial \zeta^2}w + \frac{\mathrm{i}}{\bar{\sigma}Re}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^4} - 2\bar{\sigma}^2\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} + \bar{\sigma}^4 w\right) \\ &+ \bar{\delta}^1\left\{\left(2\frac{\partial V_y}{\partial \zeta}\cos\varepsilon - \frac{\partial U}{\partial \zeta}\sin\varepsilon\right)v + [2V_y \cos\varepsilon - (U_x + c)\sin\varepsilon - 2]\frac{\mathrm{d}y}{\mathrm{d}\zeta}\right. \\ &- \frac{\mathrm{i}}{\bar{\sigma}}\left\{(V_y \sin\varepsilon + c\cos\varepsilon)\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} + \frac{\partial V_y}{\partial \zeta}\sin\varepsilon\frac{\mathrm{d}w}{\mathrm{d}\zeta} - \sigma^2 U_x \cos\varepsilon w\right. \\ &\left.\left.+ \frac{1}{k}\frac{\partial}{\partial \zeta}\left[W_z\left(\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} - \bar{\sigma}^2 w\right)\right]\right\}\right\}\end{aligned} \tag{8b}$$

In equation (8) we have also put

$$\bar{\sigma} = k\sigma; \quad \overline{Re} = k\,Re; \quad \bar{\delta}^1 = k\delta^1$$

NUMERICAL

We seek solutions of equation (8) in the weak form

$$v(z) = \sum_{i=1}^{N} v_i B_i(z)$$

$$w(z) = \sum_{j=1}^{N} w_j B_j(z) \tag{9}$$

Here the $B_i(z)$, $1 \leqslant i \leqslant N$ are cubic B-splines defined over the partition

$$\Pi{:}0 = z_1 < z_2 < \cdots < z_{l+1} = 1 \tag{10}$$

with uniform smoothness $v_i = v = 3, 2 \leqslant i \leqslant l$ on the interior breakpoints, and a knot sequence $\{t_i\}_{i=1}^{N+4}$ given by

$$\begin{aligned} z_1 &= t_1 = t_2 = t_3 = t_4 \\ z_2 &= t_5 \\ &\ldots \\ z_l &= t_N \\ z_{l+1} &= t_{N+1} = t_{N+2} = t_{N+3} = t_{N+4} \end{aligned} \tag{11}$$

Expansions (9) can be forced to satisfy boundary conditions (7) in the strong form. This yields

$$v(\zeta) = \sum_{i=2}^{N-1} v_i B_i(\zeta) \tag{12a}$$

$$w(\zeta) = \sum_{j=3}^{N-2} w_j B_j(\zeta) \tag{12b}$$

Approximations (12) are substituted into equation (8), together with spline expansions for the basic flow.

$\{U_x, V_y, W_z\}$ is available in tabulated form[8] Nevertheless we re-solved the Karman problem and obtained the basic flow via Hammings modified predictor–corrector method from an initial value problem. The spline fit was then performed with $30 \leqslant N \leqslant 75$, the number of splines depending on the conditions of the problem.

Applying (12), and Galerkin's method, differential equations (8a) and (8b) are replaced by the equivalent algebraic system

$$|(AR + \mathrm{i}AI) - c(BR + \mathrm{i}BI)| = 0 \tag{13}$$

for real matrices AR, AI, BR and BI.

The basic flow is stable if no eigenvalue exists with $\mathrm{Im}(c) > 0$. It is marginally or neutrally stable if there exists one eigenvalue with $\mathrm{Im}(c) = 0$ and for all other eigenvalues $\mathrm{Im}(c) < 0$. It is unstable if at least one eigenvalue exists with $\mathrm{Im(c)} > 0$.

To investigate the accuracy of Galerkin's method with B-splines for stability calculations, we performed numerical studies of plane Poiseuille flow. The choice for selecting this flow is obvious. Our equations are easily reducible to the Orr–Sommerfeld equation (13) for pressure flow between

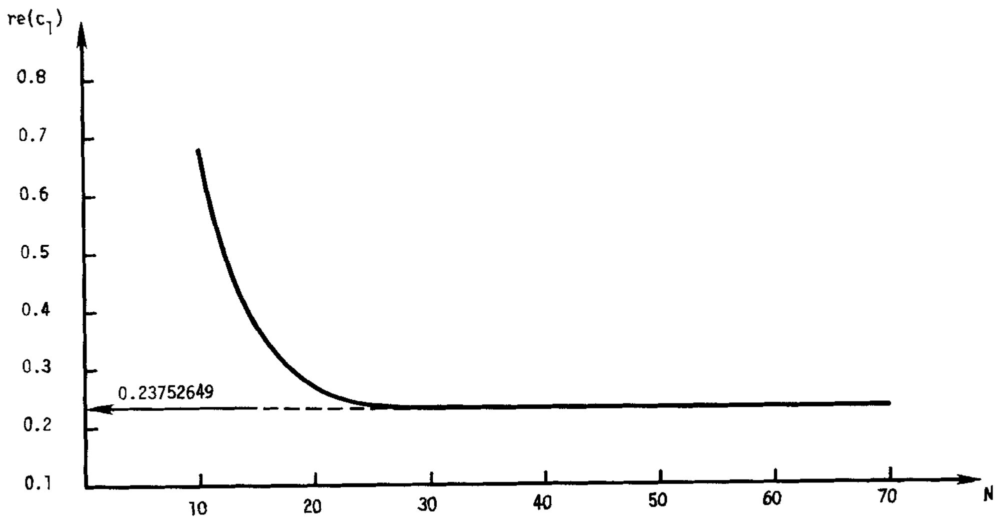

Figure 1. Plane Poiseuille flow ($Re = 20,000$, $\sigma = 2{\cdot}0$;—, present work; $\mathrm{Re}(c_1) = 0{\cdot}23752649$, Orszag[4])

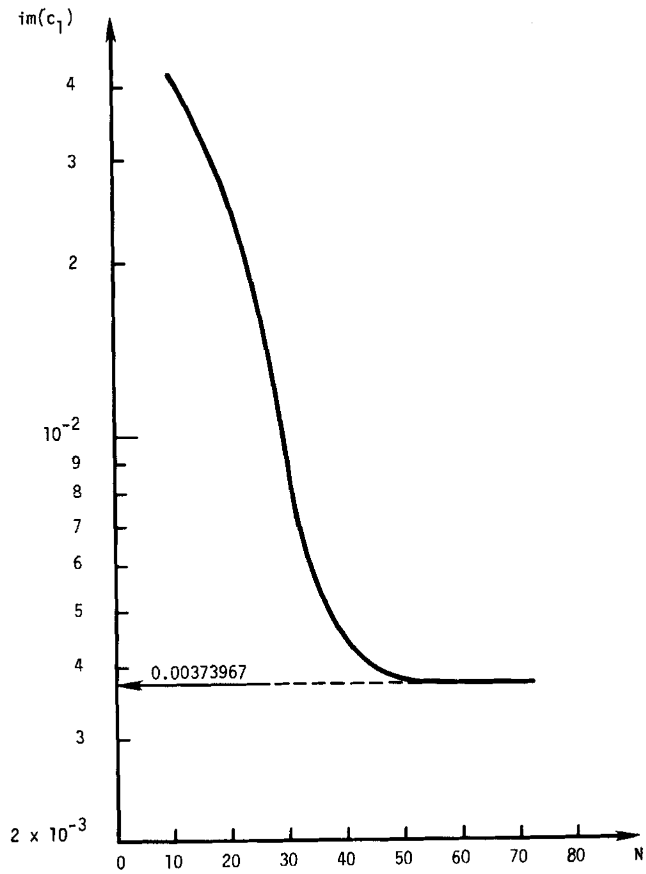

Figure 2. Plane Poiseuille flow ($Re = 20,000$, $\sigma = 2{\cdot}0$;—, present work; $\mathrm{Im}(c_i) = 0{\cdot}00373967$, Orszag[4])

parallel flat plates, and the stability of this flow has been studied extensively. Grosch and Salwen[9] used expansions in the eigenfunctions of the operator $(d^2/d\zeta^2 - \sigma^2)$ and at $Re = 20{,}000$ and $\sigma = 2\cdot0$ calculated the first eigenvalue as $c_1 = 0\cdot237413 + 0\cdot002681i$, using expansions involving up 50 symmetric eigenmodes. Orszag[4] employed Chebyshev polynomial expansion and obtained $c_1 = 0\cdot23752649 + 0\cdot00373967i$ on a CDC 6600 computer (15 significant digits in single precision). To date Orszag's calculations have the best accuracy, but the method is tiresome to apply for more complicated equations, and it requires high order machine capabilities. Furthermore, its extension to problems governed by partial differential equations is not obvious. Our result for $Re = 20{,}000$ and $\sigma = 2\cdot0$ at $N = 70$ is $c_1 = 0\cdot237394 + 0\cdot00373133i$, obtained on a PDP 10 (8 significant digits in single precision). Figures 1 and 2 compare the real and the imaginary part, respectively, of the first eigenvalue as a function of the number of splines, with the value given by Orszag. Our result of $Re_c = 11{,}537\cdot09$ at $N = 70$ is in error only by $-0\cdot06$ per cent, relative to the value calculated by Orszag.[4]

As a second step in this study of the accuracy of the present method we examined stability of flow over a rotating disk when both curvature and Coriolis effects are neglected. To calculate stability of this flow, Brown[2] solved a boundary value problem by matching the velocity of inner to outer layer. Kobayashi *et al.*[3] recalculated this problem, using a predictor–corrector routine in shooting from 'infinity', designated here z_∞, to the disk. It is crucial in these calculations to arrive at an adequate positioning of 'infinity'. As k increases, the magnitude of the critical Reynolds number becomes asymptotic. According to Figure 3 the effect of k greatly diminishes for $k > 10$. In all subsequent calculations we used $k = 10\cdot7$. We approach Re_c from below, but seem to converge to a value somewhat higher (242·84) than predicted either by Brown (233) or by Kobayshi (238). It is difficult to make detailed comparison because neither of the above referenced authors provides sufficient details of their calculations.

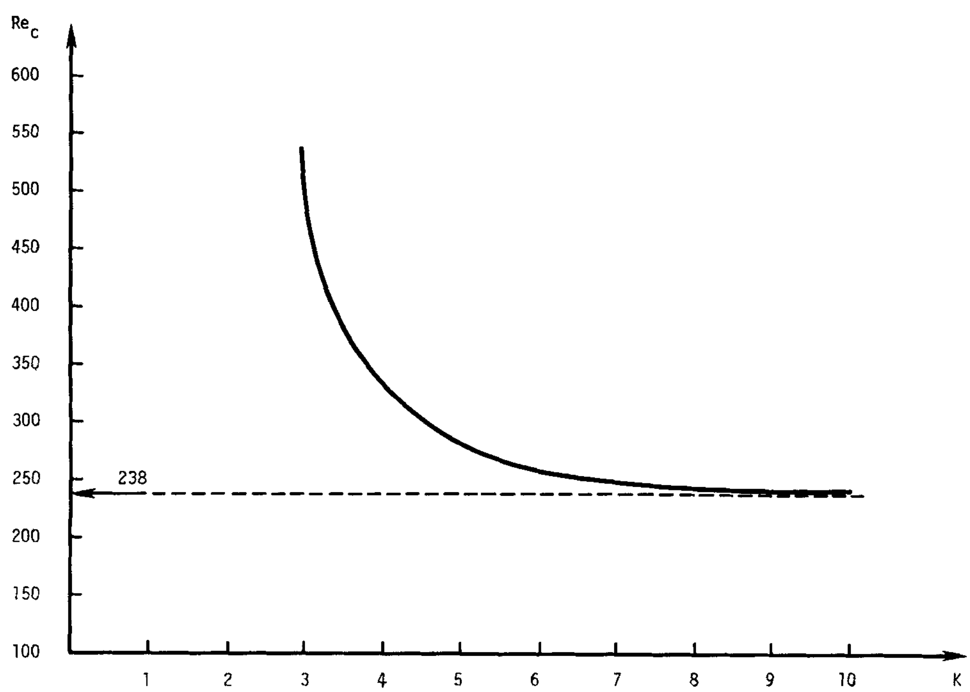

Figure 3. Flow over rotating disk, Orr–Sommerfeld equation (14). (—, present work; $Re_c = 238$, Kobayashi *et al.*[3])

DISCUSSION AND RESULTS

If in equation (8) we put $k \to \infty$, the terms containing the axial velocity W_z and its derivatives will vanish. To reduce our equation (8) to the equations used by Kobayashi *et al.*[3], we in addition must disregard variations of U_x, and V_y along $\zeta =$ constant planes.

We have re-solved Kobayashi's model. The critical Reynolds number Re_c varies with the number of splines N. Convergence is from below with increasing N, as appropriate, but convergence is to a value (335·76) somewhat higher than the $Re_c = 332$ calculated by Kobayashi. The difference amounts to 1·12 per cent. This is almost two orders of magnitude higher than the discrepancy we have with Orszag's result for plane Poiseuille flow. We are reluctant to accept that the present calculations are this much off the exact value. Rigorous comparison with the calculated data of Kobayashi is again difficult owing to lack of sufficient information.

We have indicated in Figure 3 the effect of imposing the far away boundary conditions at $z_\infty = k$ for varying k. There we concluded that the effect of k becomes negligible for $k > 10{\cdot}7$. The solution of the Karman problem also shows that at $z(\omega/v)^{1/2} = 13{\cdot}6$, which corresponds to $k = 10{\cdot}7$, $F = 3 \times 10^{-5}$, $G = 5{\cdot}3 \times 10^{-4}$ and $H = -0{\cdot}88565$. These values are to be contrasted with the theoretical boundary conditions at infinity of $F = G = 0{\cdot}0$ and $H = -0{\cdot}886$.

To find the critical vortex angle ε, equation (8) was used to plot stability diagrams at various constant values of ε. The minimum Reynolds number for a given ε was estimated, and plotted against ε; this Figure suggests $\varepsilon \cong 14°$ for critical vortex angle.

Stability diagrams were calculated based on Brown's model, Kobayashi's model, and the present model, equation (8). The results are displayed in Figure 4.

Considering that $Re = 1{\cdot}271\, R_E^{1/2}$, where $R_E = a^2\omega/v$, and that the number of vortices is given by $n = \sigma R_E^{1/2} \sin \varepsilon / 1{\cdot}271$, the present model yield at criticality

$$R_E = 8{\cdot}0128 \times 10^4$$
$$\varepsilon \cong 14°$$
$$n \cong 26$$

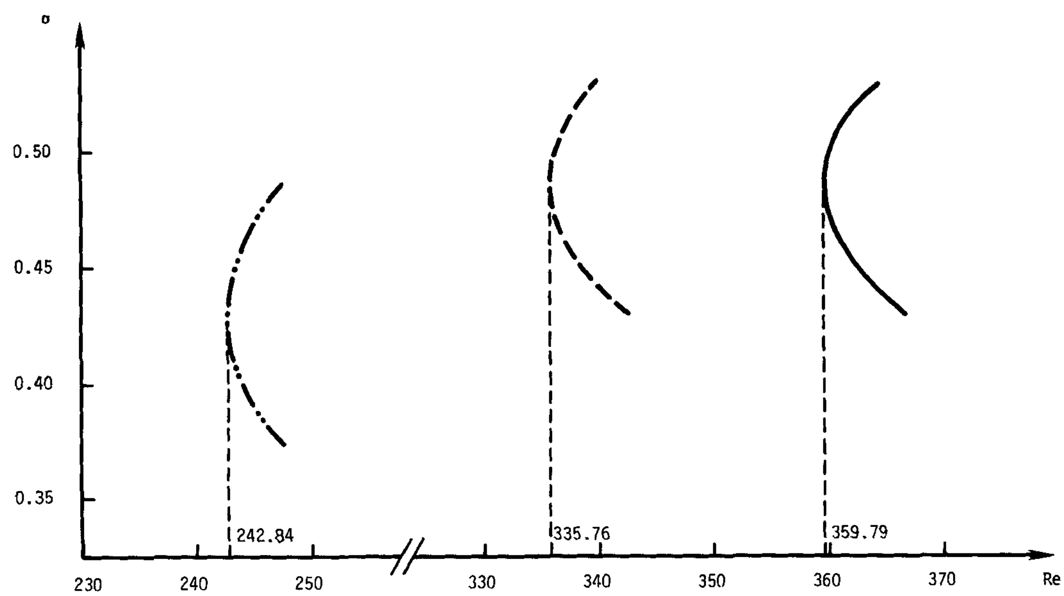

Figure 4. Stability diagrams for flow over finite, rotating disk. (- -, Brown's model, equation (14); - - - -, Kobayashi's model, equation (15); ——, present model, equation (12))

These values are to be contrasted with the experimental data of Kobayashi *et al.*[3] of $R_E = 8{\cdot}8 \times 10^4$, $\varepsilon = 13°–15°$ and $n = 26–33$.

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